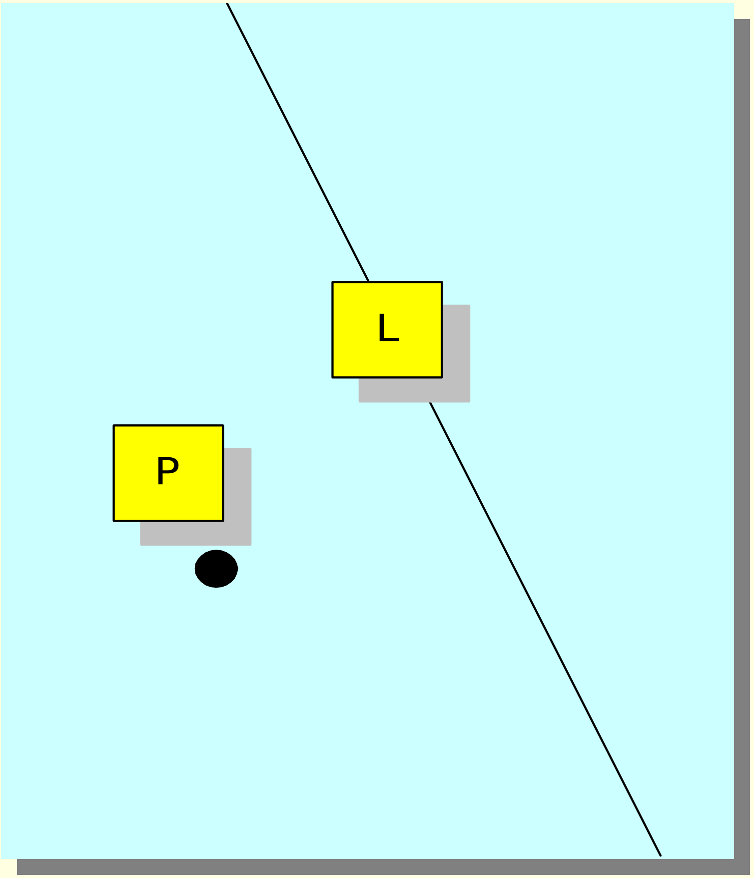
Basic Homogeneous Geometry

The Problems

2DH Points and Lines

$$\mathbf{P} = \begin{bmatrix} x & y & w \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

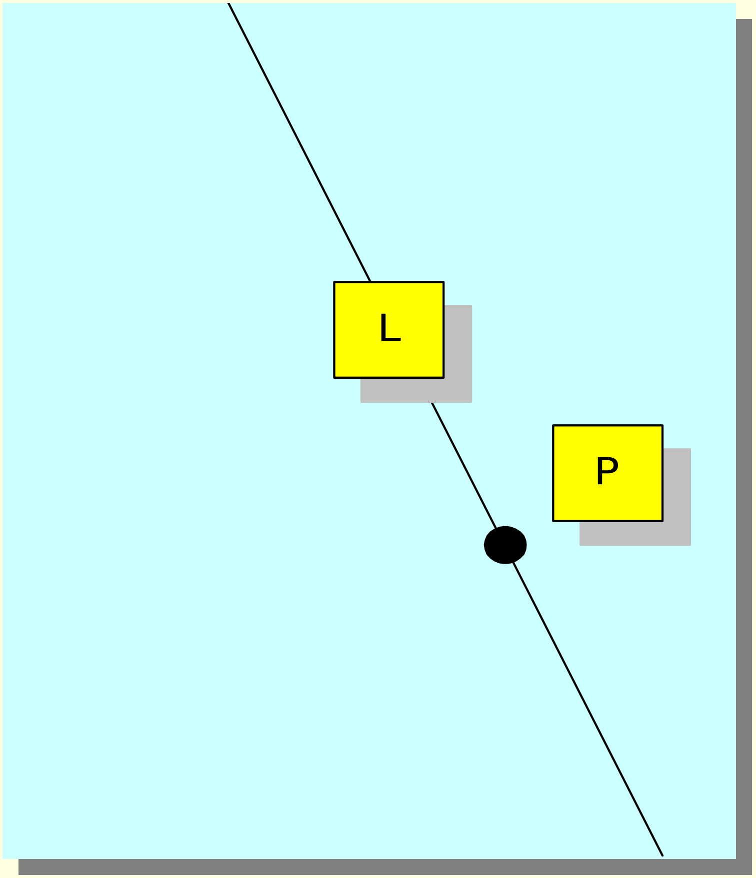


2DH Point on a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{L} = 0$$



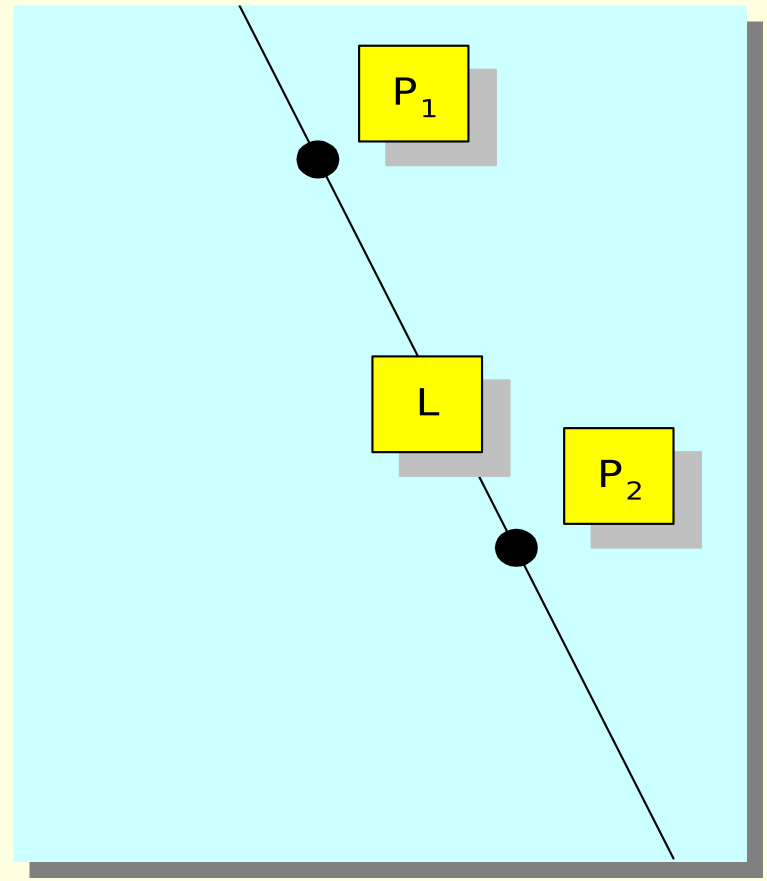
2DH Two Points Make A Line

$$\mathbf{P}_1 \times \mathbf{P}_2 = \mathbf{L}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \times \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

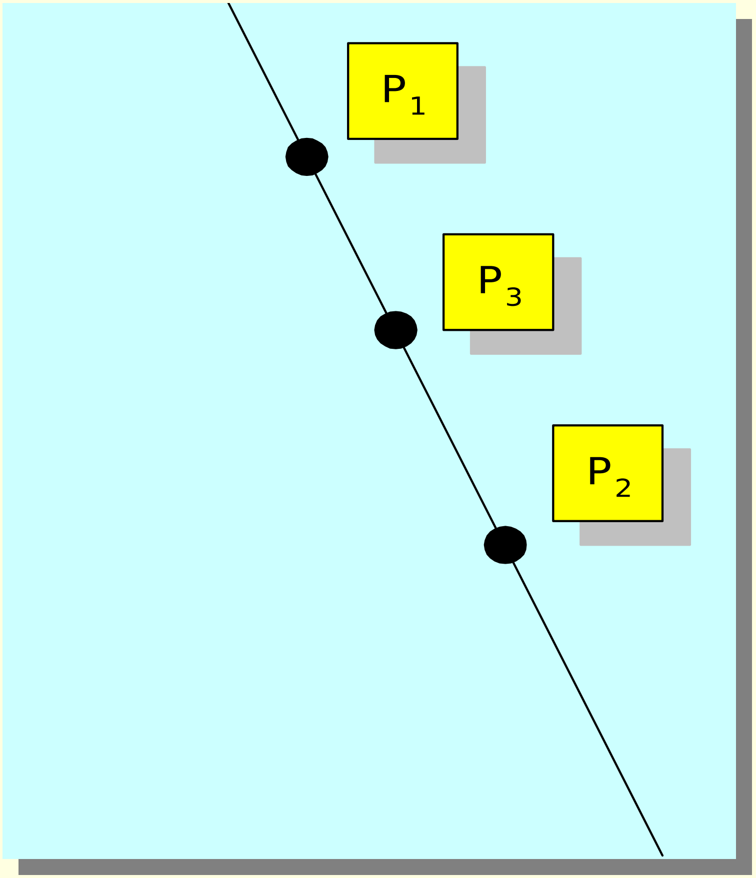
$$a = \det \begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix}$$

$$b = \det \begin{bmatrix} w_1 & x_1 \\ w_2 & x_2 \end{bmatrix} \quad c = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$



2DH Three Collinear Points

$$\mathbf{P}_1 \times \mathbf{P}_2 \cdot \mathbf{P}_3 = 0$$



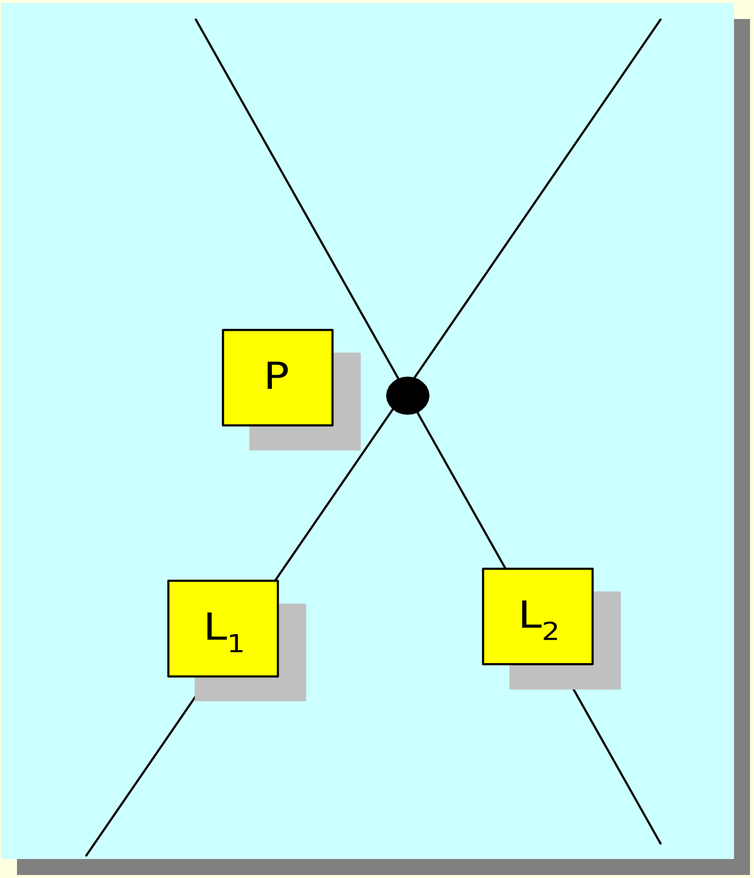
2DH Two Lines Make A Point

$$\mathbf{L}_1 \times \mathbf{L}_2 = \mathbf{P}$$

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

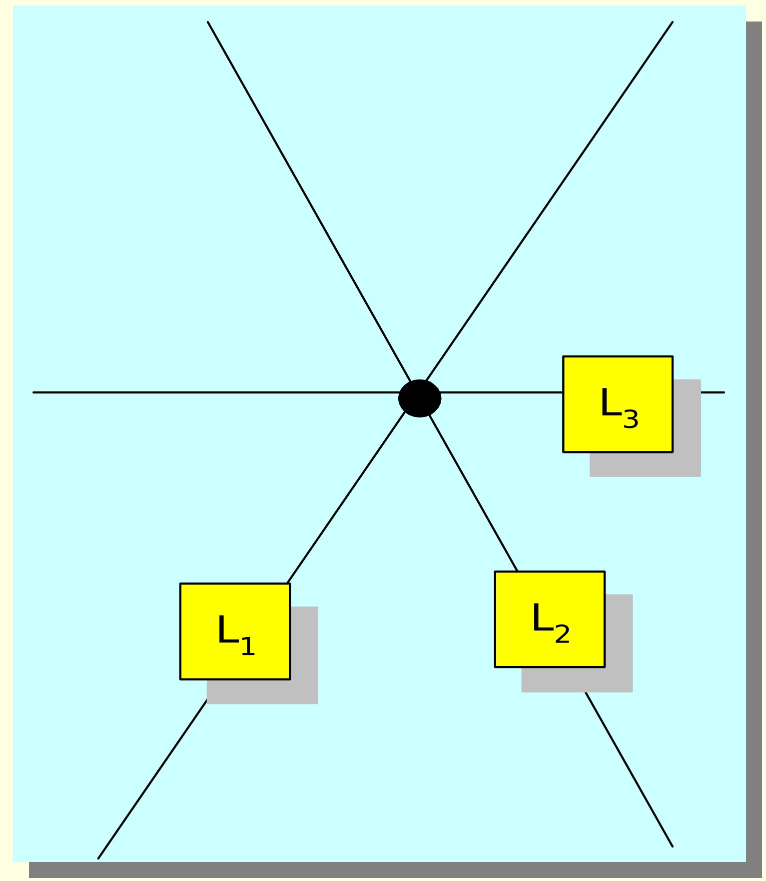
$$x = \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \qquad y = \det \begin{bmatrix} c_1 & c_2 \\ a_1 & a_2 \end{bmatrix}$$

$$z = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$



2DH Three CoPointar Lines

$$\mathbf{L}_1 \times \mathbf{L}_2 \cdot \mathbf{L}_3 = 0$$



2DH Transforming Points

$$\mathbf{PT} = \hat{\mathbf{P}}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

2DH Transforming Lines

$$\mathbf{P} \cdot \mathbf{L} = 0 \Leftrightarrow \hat{\mathbf{P}} \cdot \hat{\mathbf{L}} = 0$$

$$\begin{aligned}
\mathbf{P} \cdot \mathbf{L} &= \mathbf{P}\left(\mathbf{T}\mathbf{T}^{-1}\right)\mathbf{L} \\
&= \left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^{-1}\mathbf{L}\right) \\
&= \hat{\mathbf{P}}\left(\mathbf{T}^{-1}\mathbf{L}\right)
\end{aligned}$$

$$\boxed{\mathbf{T}^{-1}\mathbf{L} = \hat{\mathbf{L}}}$$

2DH Matrix Adjoint

$$\mathbf{T} = \begin{bmatrix} \cdots & R_1 & \cdots \\ \cdots & R_2 & \cdots \\ \cdots & R_3 & \cdots \end{bmatrix}$$

$$\mathbf{T}^* = \begin{bmatrix} \vdots & \vdots & \vdots \\ R_2 \times R_3 & R_3 \times R_1 & R_1 \times R_2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{T}\mathbf{T}^* = \left(\det \mathbf{T}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}^* = \left(\det \mathbf{T}\right) \mathbf{T}^{-1}$$

2DH Transforming Points and Lines

$$\mathbf{P}\mathbf{T} = \hat{\mathbf{P}} \qquad [x \quad y \quad w]\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = [\hat{x} \quad \hat{y} \quad \hat{w}]$$

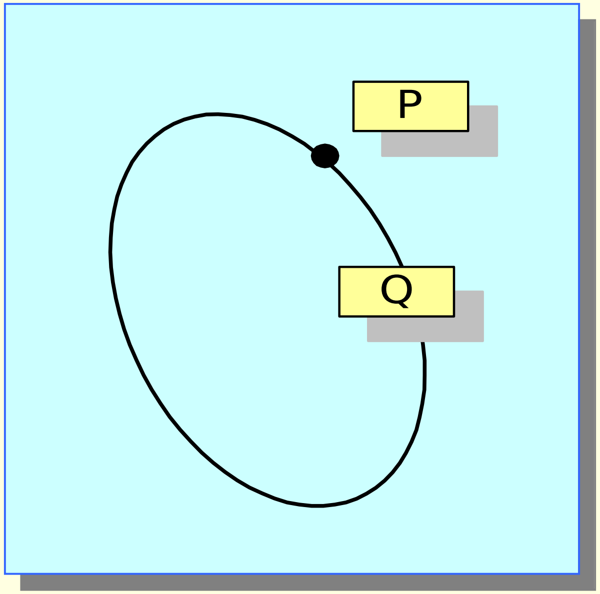
$$\mathbf{T}^{*}\mathbf{L} = \hat{\mathbf{L}} \qquad \begin{bmatrix} T^{*}_{11} & T^{*}_{12} & T^{*}_{13} \\ T^{*}_{21} & T^{*}_{22} & T^{*}_{23} \\ T^{*}_{31} & T^{*}_{32} & T^{*}_{33} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}$$

2DH Point on Quadratic Curve

$$Ax^2 + 2Bxy + 2Cxw$$
$$+ Dy^2 + 2Eyw$$
$$+ Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$



2DH Transforming a Quadratic

$$\mathbf{P}\mathbf{Q}\mathbf{P}^T = 0 \;\; \hat{U} \;\; \hat{\mathbf{P}}\hat{\mathbf{Q}}\hat{\mathbf{P}}^T = 0$$

$$\mathbf{P}\mathbf{Q}\mathbf{P}^T = d^2 \; \mathbf{P}\left(\mathbf{T}\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}\mathbf{T}^*\right)^T \mathbf{P}^T$$

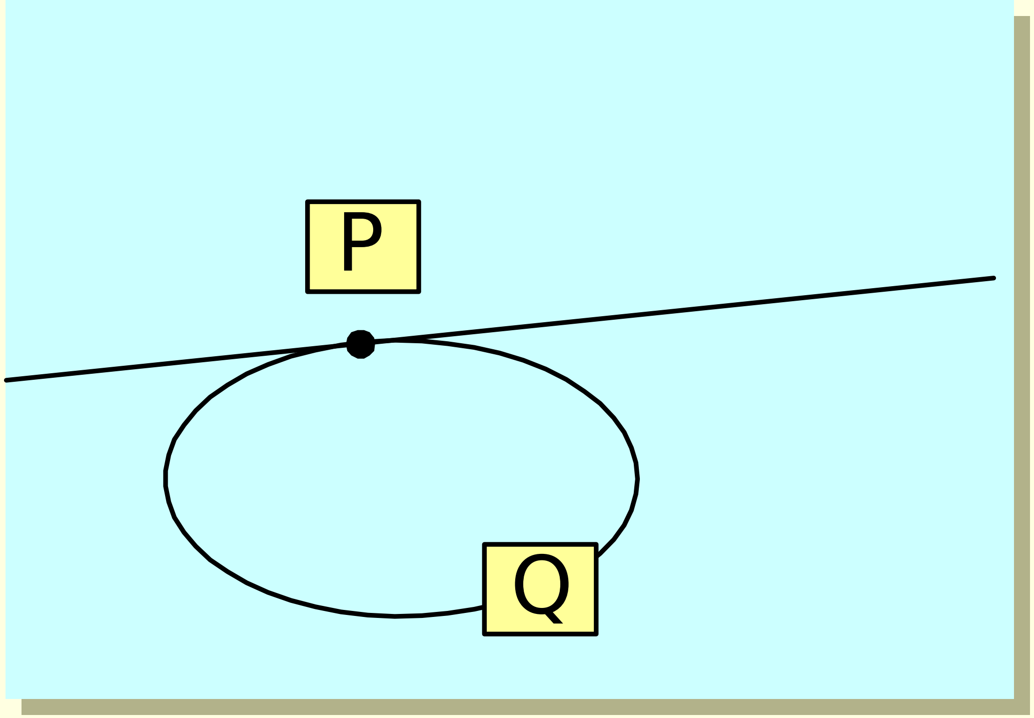
$$= d^2 \left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T}\right)\left(\mathbf{P}\mathbf{T}\right)^T$$

$$= d^2 \; \hat{\mathbf{P}}\left(\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T}\right)\hat{\mathbf{P}}^T$$

$$\boxed{\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T} = \hat{\mathbf{Q}}}$$

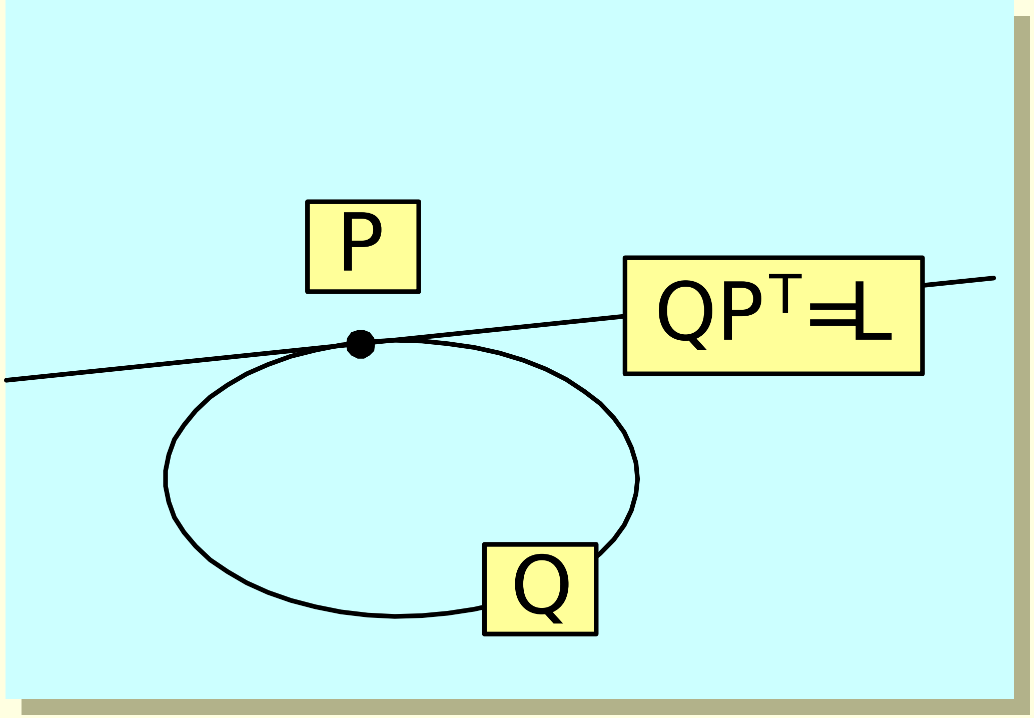
2DH Tangent at a Point

$$
\begin{aligned}
0 &= \mathbf{P}\mathbf{Q}\mathbf{P}^T \\
&= \mathbf{P} \times \left(\mathbf{Q}\mathbf{P}^T\right) \\
&= \mathbf{P} \times \mathbf{L}
\end{aligned}
$$



2DH Tangent at a Point

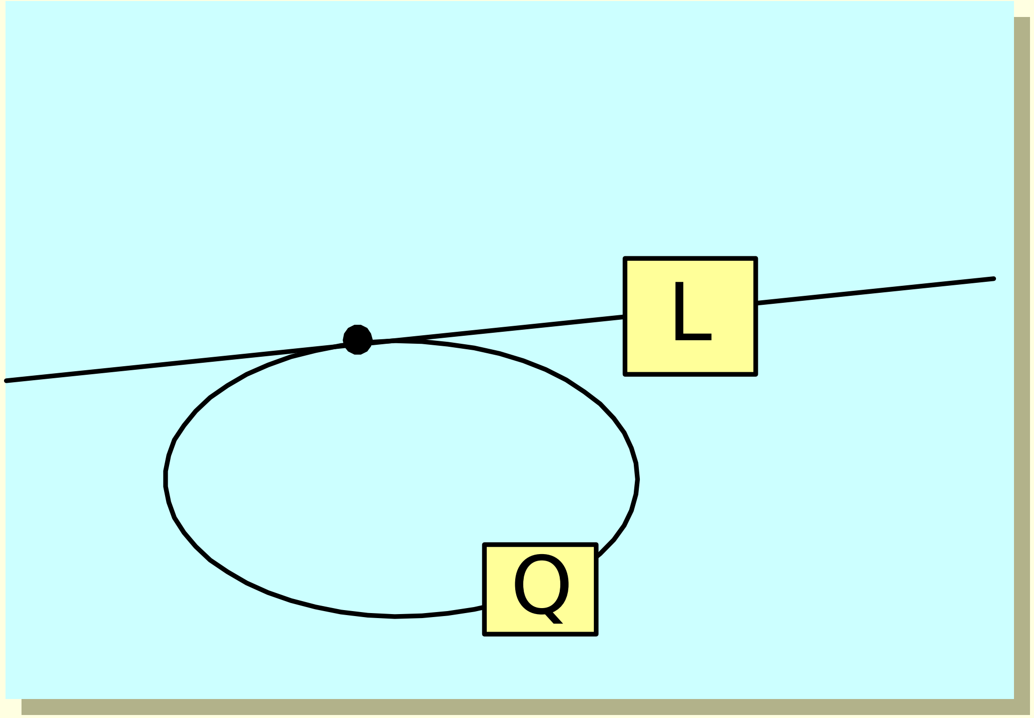
$$
\begin{aligned}
0 &= \mathbf{P}\mathbf{Q}\mathbf{P}^T \\
&= \mathbf{P} \times \left(\mathbf{Q}\mathbf{P}^T\right) \\
&= \mathbf{P} \times \mathbf{L}
\end{aligned}
$$



2DH Line Tangent to Quadric

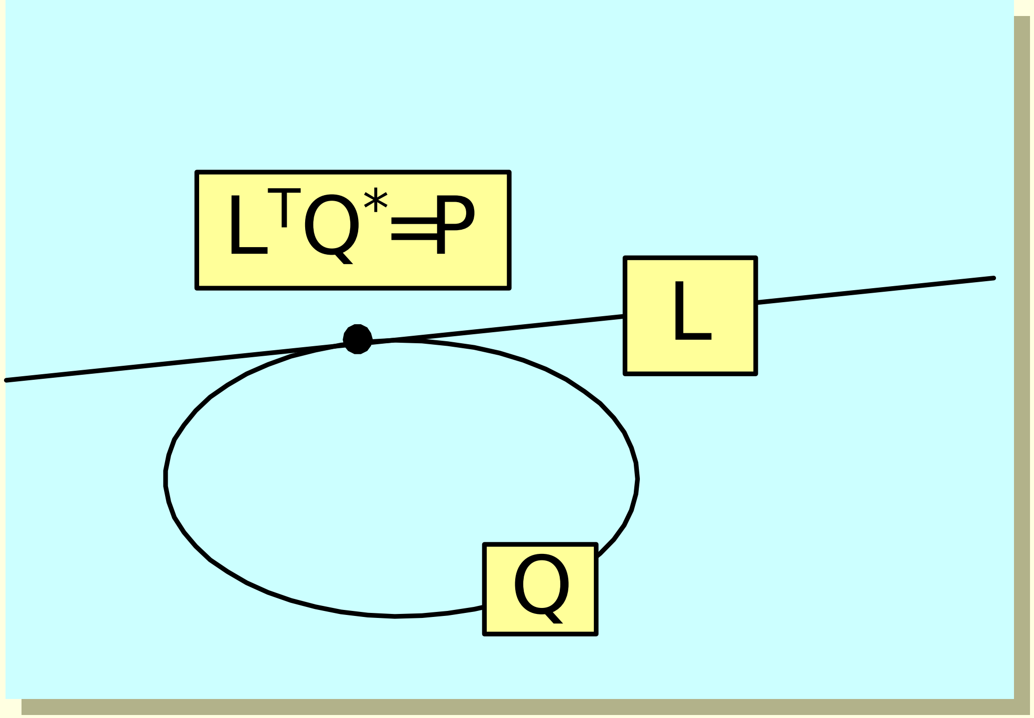
$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$

$$= \left(\mathbf{L}^T \mathbf{Q}^* \right) \mathbf{L}$$



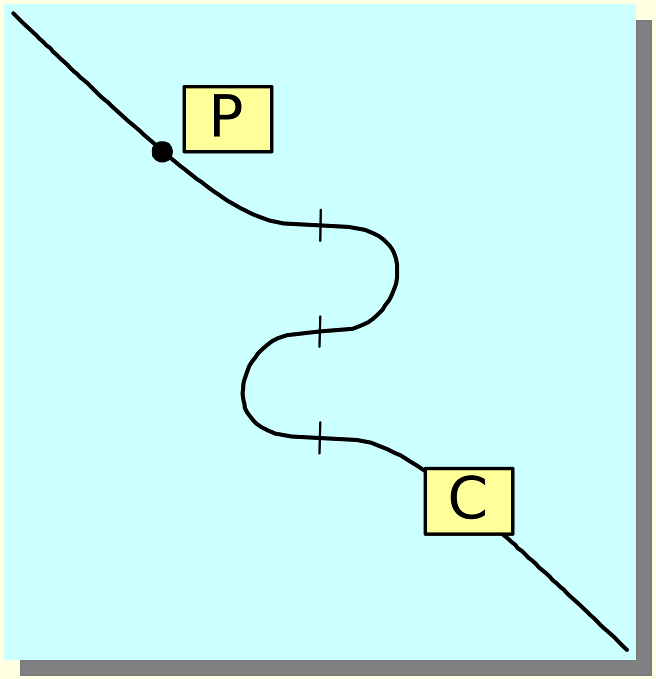
2DH Line Tangent to Quadric

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$
$$= \left(\mathbf{L}^T \mathbf{Q}^* \right) \mathbf{L}$$



2DH Point on Cubic Curve

$$Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gyw^2$$
$$+2Hxw^2 + 3Jyw^2$$
$$+Kw^2 = 0$$



2DH Cubic Curve

$$
\begin{aligned}
Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3 \\
+3Ex^2w + 6Fxyw + 3Gyw^2 \\
+2Hxw^2 + 3Jyw^2 \\
+Kw^2 = 0
\end{aligned}
$$

$$
\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$

$$
\left\{ \mathbf{PCP}^T \right\} \mathbf{P}^T = 0
$$

2DH Curves of Various Orders

$$\mathrm{L} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}$$

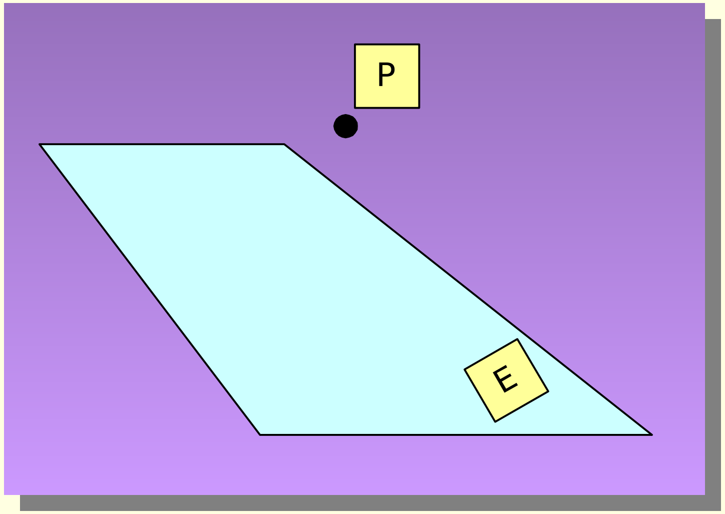
$$\mathbf{C} = \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix}$$

Now 3D (Homogeneous)

3DH Points and Planes

$$P = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

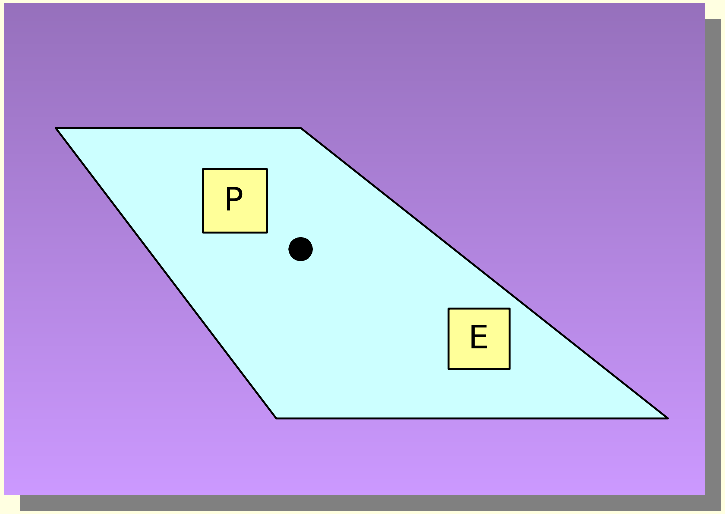
$$E = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



3DH Point on Plane

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{E} = 0$$



3DH Transformations

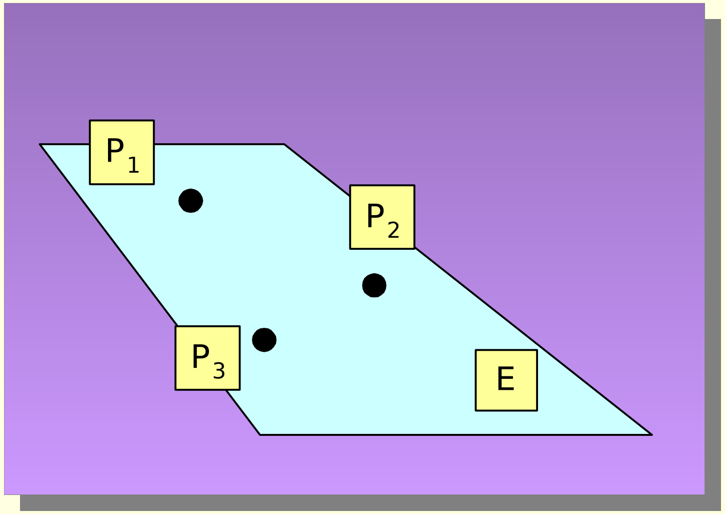
$$\mathbf{P}\mathbf{T} = \mathbf{P}'$$

$$\mathbf{T}^{*}\mathbf{E} = \mathbf{E}'$$

3DH Plane thru 3 Points

$$cross\left(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\right) = \mathbf{E}$$

$$cross\begin{pmatrix} [x_1 & y_1 & z_1 & w_1], \\ [x_2 & y_2 & z_2 & w_2], \\ [x_3 & y_3 & z_3 & w_3] \end{pmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

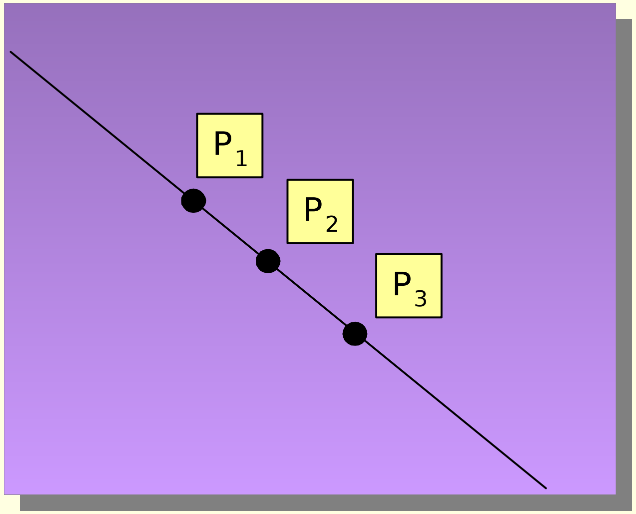


$$a = \det\begin{bmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{bmatrix}, \quad b = -\det\begin{bmatrix} x_1 & z_1 & w_1 \\ x_2 & z_2 & w_2 \\ x_3 & z_3 & w_3 \end{bmatrix}$$

$$c = \det\begin{bmatrix} y_1 & y_1 & w_1 \\ y_2 & y_2 & w_2 \\ y_3 & y_3 & w_3 \end{bmatrix}, \quad d = -\det\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

3DH Three Collinear points

$$cross\left(\begin{matrix}[x_1 & y_1 & z_1 & w_1],\\ [x_2 & y_2 & z_2 & w_2],\\ [x_3 & y_3 & z_3 & w_3]\end{matrix}\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$



3DH Rewrite Equation

$$cross\begin{pmatrix} [x_1 & y_1 & z_1 & w_1], \\ [x_2 & y_2 & z_2 & w_2], \\ [x_3 & y_3 & z_3 & w_3] \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = \det\begin{bmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{bmatrix}$$

$$0 = y_3 \det\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} - z_3 \det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} + w_3 \det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & \det\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} & -\det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} & \det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix}$$

3DH Separate P_1P_2 from P_3

$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

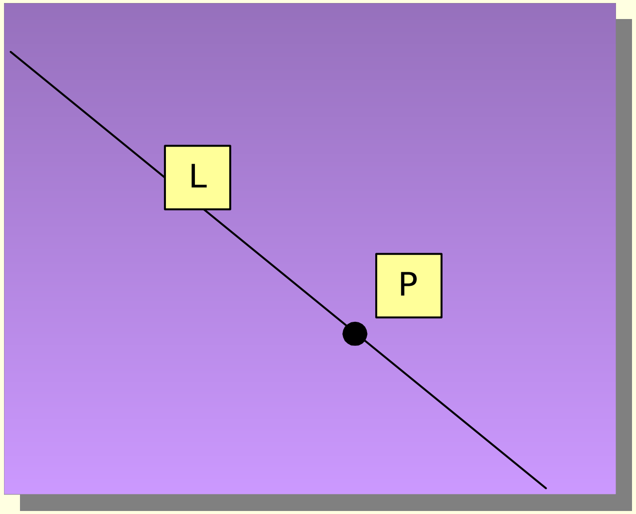
$$p = \det \begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} \qquad q = \det \begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} \qquad r = \det \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

$$s = \det \begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix} \qquad t = \det \begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix} \qquad u = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$$pu - qt + sr = 0$$

3DH Point on Line

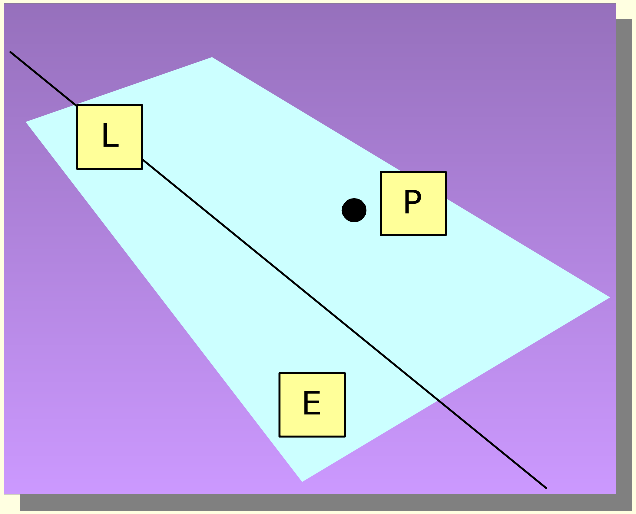
$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\mathbf{L}\mathbf{P}^T = \mathbf{0}$$

3DH Point not on Line = Plane

$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



$$\mathbf{L}\mathbf{P}^T = \mathbf{E}$$

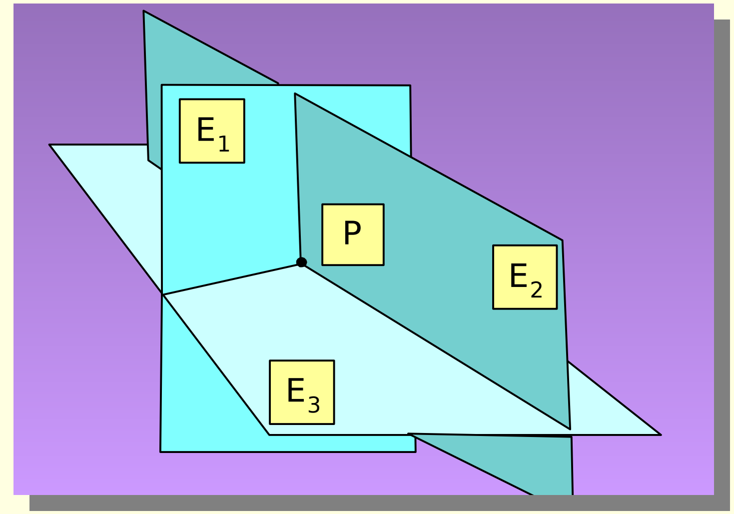
3DH Transforming a Line

$$\mathbf{L}\mathbf{P}^T = \mathbf{E} \Leftrightarrow \mathbf{L}'\mathbf{P}'^T = \mathbf{E}'$$

$$\boxed{\mathbf{T}^*\mathbf{L}\left(\mathbf{T}^*\right)^T = \mathbf{L}'}$$

3DH Point on 3 Planes

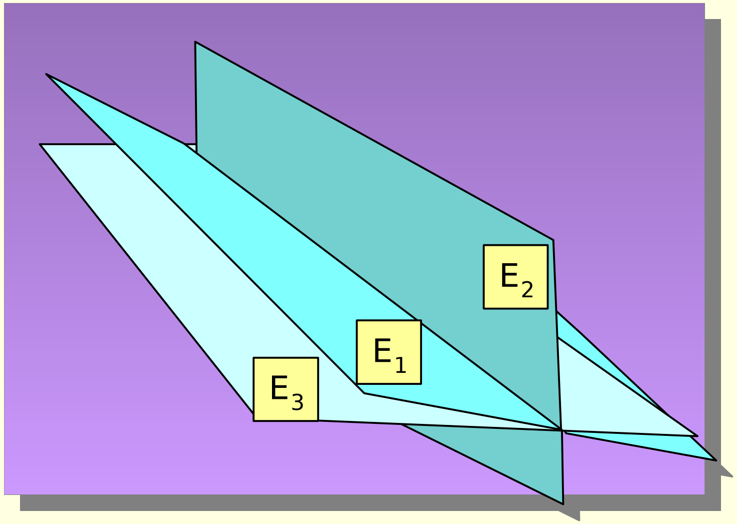
$$cross\left(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\right) = \mathbf{P}$$



$$crs4\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} \right) = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$x = \det\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad y = -\det\begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \quad z = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad w = -\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

3DH Three Collinear Planes



$$crs4\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

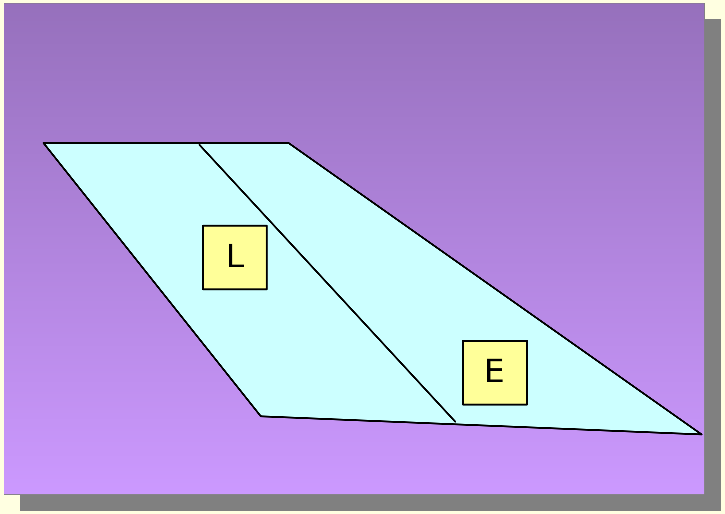
3DH Separate E_1E_2 from E_3

$$\begin{bmatrix} a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e = \det \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \quad f = \det \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \quad g = \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$h = \det \begin{bmatrix} a_1 & a_2 \\ d_1 & d_2 \end{bmatrix} \quad j = \det \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \quad k = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

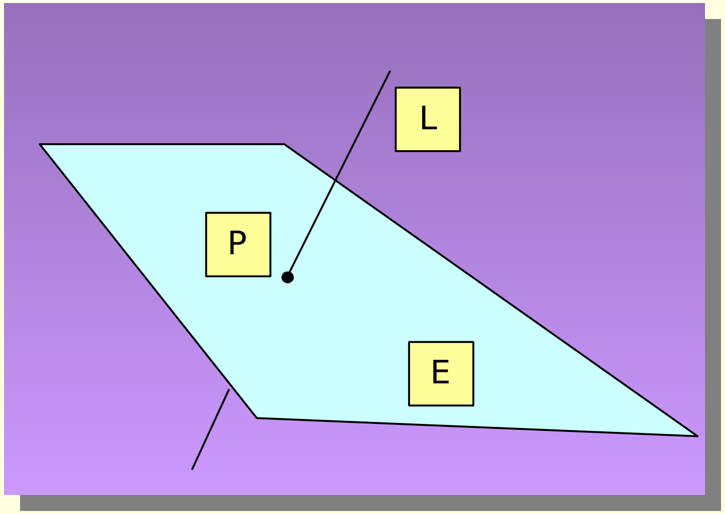
3DH Line embedded in Plane



$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^T \mathbf{K} = \mathbf{0}$$

3DH Line Not in Plane = Point



$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$\mathbf{E}^T \mathbf{K} = \mathbf{P}$$

3DH Two Forms of Line

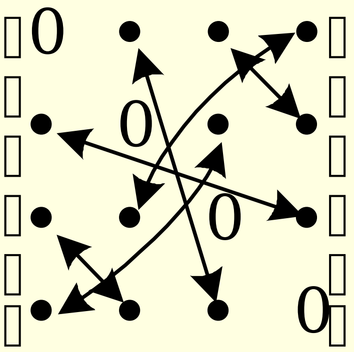
$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} = \mathbf{L} \qquad \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \mathbf{K}$$

$$\mathbf{L}\mathbf{P}^T = \mathbf{E} \qquad \mathbf{E}^T\mathbf{K} = \mathbf{P}$$

3DH Converting Between Two Forms of Line

$$\mathbf{L} = \begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \mathbf{K} = \begin{bmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{bmatrix}$$



$$e = -u, \quad f = t, \quad g = -s$$

$$h = -r, \quad j = q, \quad k = -p$$

Two Problems

Rows vs. Columns

$$\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} x & y & z & w \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix}$$

More than Two Indices

$$\begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix}$$